doi:10.1006/jmaa.2000.7022, available online at http://www.idealibrary.com on IDEAL®

Existence of Solutions of Nonlinear Neutral Integrodifferential Equations in Banach Spaces

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Submitted by William F. Ames

Received April 24, 2000

In this paper we prove the existence of mild solutions of a nonlinear neutral integrodifferential equation in a Banach space. The results are obtained by using the Schaefer fixed point theorem. As an application the controllability problem for the neutral system is discussed. © 2000 Academic Press

Key Words: existence of solution; neutral integrodifferential equation; Schaefer's fixed point theorem; controllability.

1. INTRODUCTION

The theory of neutral delay differential equations has been extensively studied in the literature [1, 2, 7–9]. Hernandez and Henriquez [5] obtained some existence results for neutral functional differential equations in Banach spaces, and in [6] they established the existence of periodic solutions for the same kind of equations. In both papers Hernandez and Henriquez used semigroup theory and the Sadovski fixed point principle. Recently Balachandran and Sakthivel [3] studied the existence of solutions for neutral functional integrodifferential equations in Banach spaces.

0022-247X/00 $35.00

The purpose of this paper is to prove the existence of mild solutions for nonlinear neutral integrodifferential equations of the form

$$\frac{d}{dt}[x(t) - g(t, x_t)] = Ax(t) + f\left(t, x_t, \int_0^t h(t, s, x_s)ds\right),$$
$$t \in J = [0, b], \qquad x_0 = \phi, \qquad \text{on } [-r, 0], \tag{1}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in a Banach space X, where $h: J \times J \times C \to X$, $f: J \times C \times X \to X$, and $g: J \times C \to X$ are continuous functions. Here $C = C([-r, 0], X)$ is the Banach space of all continuous functions $\phi: [-r, 0] \to X$ endowed with the norm $\|\phi\| = \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}$. Also, for $x \in C([-r, b], X)$ we have $x_t \in C$ for $t \in [0, b]$, $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$.

2. PRELIMINARIES

In order to define the concept of mild solution for (1), by comparison with the abstract Cauchy problem

$$x'(t) = Ax(t) + f(t)$$

whose properties are well known [10], we associate problem (1) to the integral equation

$$x(t) = T(t)[\phi(0) - g(0, \phi)] + g(t, x_t) + \int_0^t AT(t-s)g(s, x_s)ds$$
$$+ \int_0^t T(t-s)f\left(s, x_s, \int_0^s h(s, \tau, x_\tau)d\tau\right)ds, \qquad t \in [0, b]. \tag{2}$$

DEFINITION 2.1. A function $x: (-r, b) \to X$, $b > 0$ is called a mild solution of the Cauchy problem (1) if $x_0 = \phi$, the restriction of $x(\cdot)$ to the interval $[0, b)$, is continuous, if for each $0 \leq t < b$ the function $AT(t-s)g(s, x_s)$, $s \in [0, t)$, is integrable, and if the integral equation (2) is satisfied.

We need the following fixed point theorem due to Schaefer [11].

THEOREM 2.1. *Let E be a normed linear space. Let $F: E \to E$ be a completely continuous operator, that is, it is continuous and the image of any bounded set is contained in a compact set and let*

$$\zeta(F) = \{x \in E: x = \lambda Fx \quad \text{for some } 0 < \lambda < 1\}.$$

Then either $\zeta(F)$ is unbounded or F has a fixed point.

Assume that the following hold:

(i) A is the infinitesimal generator of a compact semigroup of bounded linear operators $T(t)$ in X such that

$$|T(t)| \leq M_1, \qquad \text{for some } M_1 \geq 1 \quad \text{and} \quad |AT(t)| \leq M_2, \quad M_2 > 0.$$

(ii) For each $(t, s) \in J \times J$, the function $h(t, s, \cdot): C \to X$ is continuous, and for each $x \in C$ the function $h(\cdot, \cdot, x): J \times J \to X$ is strongly measurable.

(iii) For each $t \in J$ the function $f(t, \cdot, \cdot): C \times X \to X$ is continuous, and for each $(x, y) \in C \times X$ the function $f(\cdot, x, y): J \to X$ is strongly measurable.

(iv) For every positive integer k there exists $\alpha_k \in L^1(0, b)$ such that

$$\sup_{\|x\|, |y| \leq k} |f(t, x, y)| \leq \alpha_k(t), \qquad \text{for } t \in J \text{ a.e.}$$

(v) The function $g: J \times C \to X$ is completely continuous, and for any bounded set Q in $C([-r, b], X)$, the set $\{t \to g(t, x_t): x \in Q\}$ is equicontinuous in $C([0, b], X)$.

(vi) There exist constants $c_1 < 1$ and $c_2 > 0$ such that

$$|g(t, \phi)| \leq c_1 \|\phi\| + c_2, \qquad t \in J, \quad \phi \in C.$$

(vii) There exists an integrable function $m: [0, b] \to [0, \infty)$ and a constant $\alpha > 0$ such that

$$|h(t, s, x)| \leq \alpha m(s) \Omega_0(\|x\|), \qquad 0 \leq s < t \leq b, \quad x \in C,$$

where $\Omega_0: [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function.

(viii) There exists an integrable function $p: [0, b] \to [0, \infty)$ such that

$$|f(t, x, y)| \leq p(t) \Omega(\|x\| + |y|), \qquad 0 \leq t \leq b, \quad x \in C, \quad y \in X,$$

where $\Omega: [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function.

(ix)

$$\int_0^b \hat{m}(s) ds < \int_c^\infty \frac{ds}{s + \Omega_0(s) + \Omega(s)},$$

where $c = 1/(1 - c_1)[M_1(\|\phi\| + c_1\|\phi\| + c_2) + c_2 + M_2 c_2 b]$ and

$$\hat{m}(t) = \max\left\{ \frac{M_2 c_1}{1 - c_1}, \frac{M_1}{1 - c_1} p(t), \quad \alpha m(t) \right\}.$$

3. MAIN RESULT

THEOREM 3.1. *If the assumptions* (i) to (ix) *are satisfied, then the problem* (1) *has a mild solution on* $[-r, b]$.

Proof Consider the Banach space $C([-r, b] : X)$ with norm

$$\|x\|_1 = \sup\{|x(t)| : -r \leq t \leq b\}.$$

To prove the existence of a mild solution of (1) we apply Schaefer's Theorem 2.1. First we obtain *a priori* bounds for the solutions of the following problem, as in [9]:

$$\frac{d}{dt}[x(t) - \lambda g(t, x_t)] = Ax(t) + \lambda f\left(t, x_t, \int_0^t h(t, s, x_s)ds\right),$$
$$\lambda \in (0, 1), \quad t \in J,$$
$$x_0 = \lambda\phi. \tag{3}$$

Let x be a mild solution of problem (3). From

$$x(t) = \lambda T(t)[\phi(0) - g(0, \phi)] + \lambda g(t, x_t) + \lambda \int_0^t AT(t-s)g(s, x_s)ds$$
$$+ \lambda \int_0^t T(t-s)f(s, x_s, \int_0^s h(s, \tau, x_\tau)d\tau)ds,$$

we have

$$|x(t)| \leq M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_1\|x_t\| + c_2 + M_2 \int_0^t (c_1\|x_s\| + c_2)ds$$
$$+ M_1 \int_0^t p(s)\Omega[\|x_s\| + \int_0^s \alpha m(\tau)\Omega_0(\|x_\tau\|)d\tau]ds. \tag{4}$$

Consider the function μ given by

$$\mu(t) = \sup\{|x(s)| : -r \leq s \leq t\}, \qquad 0 \leq t \leq b.$$

Let $t^\star \in [-r, t]$ be such that $\mu(t) = |x(t^\star)|$. If $t^\star \in [0, b]$, by inequality (4) we have

$$\mu(t) \leq M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_1\mu(t) + c_2 + M_2c_1 \int_0^{t^*} \mu(s)ds$$
$$+ M_2c_2b + M_1 \int_0^{t^*} p(s)\Omega[\mu(s) + \alpha \int_0^s m(\tau)\Omega_0(\mu(\tau))d\tau]ds$$
$$\leq M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_1\mu(t) + c_2 + M_2c_1 \int_0^t \mu(s)ds$$
$$+ M_2c_2b + M_1 \int_0^t p(s)\Omega[\mu(s) + \alpha \int_0^s m(\tau)\Omega_0(\mu(\tau))d\tau]ds,$$

or

$$\mu(t) \leq \frac{1}{1-c_1}\{M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_2 + M_2 c_2 b + M_2 c_1 \int_0^t \mu(s)ds + M_1 \int_0^t p(s)\Omega[\mu(s) + \alpha \int_0^s m(\tau)\Omega_0(\mu(\tau))d\tau]ds\}. \tag{5}$$

If $t^\star \in [-r, 0]$, then $\mu(t) = \|\phi\|$ and the inequality (5) holds since $M_1 \geq 1$.

Denoting by $v(t)$ the right-hand side of inequality (5) we have $c = v(0) = \frac{1}{1-c_1}\left\{M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_2 + M_2 c_2 b\right\}$, $\mu(t) \leq v(t)$, $0 \leq t \leq b$ and

$$\begin{aligned} v'(t) &= \frac{1}{1-c_1} M_2 c_1 \mu(t) + \frac{1}{1-c_1} M_1 p(t)\Omega\left[\mu(t) + \alpha \int_0^t m(s)\Omega_0(\mu(s))ds\right] \\ &\leq \frac{1}{1-c_1} M_2 c_1 v(t) + \frac{1}{1-c_1} M_1 p(t)\Omega\left[v(t) + \alpha \int_0^t m(s)\Omega_0(v(s))ds\right] \\ &\leq \frac{1}{1-c_1} M_2 c_1 \left\{v(t) + \frac{M_1}{M_2 c_1} p(t)\Omega\left[v(t) + \alpha \int_0^t m(s)\Omega_0(v(s))ds\right]\right\}. \end{aligned}$$

Let $w(t) = v(t) + \alpha \int_0^t m(s)\Omega_0(v(s))ds$. Then $w(0) = v(0)$, $v(t) \leq w(t)$, and

$$\begin{aligned} w'(t) &= v'(t) + \alpha m(t)\Omega_0(v(t)) \\ &\leq \frac{1}{1-c_1} M_2 c_1 \{v(t) + \frac{M_1}{M_2 c_1} p(t)\Omega(w(t))\} + \alpha m(t)\Omega_0(w(t)) \\ &\leq \frac{M_2 c_1}{1-c_1} w(t) + \frac{M_1}{1-c_1} p(t)\Omega(w(t)) + \alpha m(t)\Omega_0(w(t)) \\ &\leq \hat{m}(t)\{w(t) + \Omega_0(w(t)) + \Omega(w(t))\}. \end{aligned}$$

This implies

$$\int_{w(0)}^{w(t)} \frac{ds}{s + \Omega_0(s) + \Omega(s)} \leq \int_0^b \hat{m}(s)ds < \int_c^\infty \frac{ds}{s + \Omega_0(s) + \Omega(s)}, \qquad 0 \leq t \leq b. \tag{6}$$

Inequality (6) implies that there is a constant K such that $v(t) \leq K$, $t \in [0, b]$ and hence $\mu(t) \leq K$, $t \in [0, b]$. Since $\|x_t\| \leq \mu(t)$, for all $t \in [0, b]$, we have

$$\|x\|_1 = \sup\{|x(t)| : -r \leq t \leq b\} \leq K,$$

where K depends only on b and on the functions m and Ω_0.

Next we rewrite problem (1) as follows. For $\phi \in C$ define $\hat{\phi} \in C_b$, $C_b = C([-r, b], X)$ by

$$\hat{\phi}(t) = \begin{cases} \phi(t), & -r \leq t \leq 0, \\ T(t)\phi(0), & 0 \leq t \leq b. \end{cases}$$

If $x(t) = y(t) + \hat{\phi}(t)$, $t \in [-r, b]$, it is easy to see that y satisfies

$$y_0 = 0$$

$$y(t) = -T(t)g(0, \phi) + g(t, y_t + \hat{\phi}_t) + \int_0^t AT(t-s)g(s, y_s + \hat{\phi}_s)ds$$
$$+ \int_0^t T(t-s)f(s, y_s + \hat{\phi}_s, \int_0^s h(s, \tau, y_\tau + \hat{\phi}_\tau)d\tau)ds$$

if and only if x satisfies

$$x(t) = T(t)[\phi(0) - g(0, \phi)] + g(t, x_t) + \int_0^t AT(t-s)g(s, x_s)ds$$
$$+ \int_0^t T(t-s)f(s, x_s, \int_0^s h(s, \tau, x_\tau)d\tau)ds$$

and $x_0 = \phi$.

Define $C_b^0 = \{y \in C_b : y_0 = 0\}$ and $F\colon C_b^0 \to C_b^0$ by

$$(Fy)(t) = 0, \qquad -r \leq t \leq 0$$

$$(Fy)(t) = -T(t)g(0, \phi) + g(t, y_t + \hat{\phi}_t) + \int_0^t AT(t-s)g(s, y_s + \hat{\phi}_s)ds$$
$$+ \int_0^t T(t-s)f\left(s, y_s + \hat{\phi}_s, \int_0^s h(s, \tau, y_\tau + \hat{\phi}_\tau)d\tau\right)ds, \quad 0 \leq t \leq b.$$

It will now be shown that F is a completely continuous operator.

Let $B_k = \{y \in C_b^0 : \|y\|_1 \leq k\}$ for some $k \geq 1$. We first show that F maps B_k into an equicontinuous family. Let $y \in B_k$ and t_1, $t_2 \in [0, b]$. Then if $0 < t_1 < t_2 \leq b$,

$$|(Fy)(t_1) - (Fy)(t_2)|$$
$$\leq |T(t_1) - T(t_2)||g(0, \phi)| + |g(t_1, y_{t_1} + \hat{\phi}_{t_1}) - g(t_2, y_{t_2} + \hat{\phi}_{t_2})|$$
$$+ \left|\int_0^{t_1} A[T(t_1 - s) - T(t_2 - s)]g(s, y_s + \hat{\phi}_s)ds\right|$$

$$+\left|\int_{t_1}^{t_2} AT(t_2-s)g(s,y_s+\hat{\phi}_s)ds\right|$$

$$+\left|\int_0^{t_1}[T(t_1-s)-T(t_2-s)]f\left(s,y_s+\hat{\phi}_s,\int_0^s h(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau\right)ds\right|$$

$$+\left|\int_{t_1}^{t_2} T(t_2-s)f\left(s,y_s+\hat{\phi}_s,\int_0^s h(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau\right)ds\right|$$

$$\le |T(t_1)-T(t_2)||g(0,\phi)|+|g(t_1,y_{t_1}+\hat{\phi}_{t_1})-g(t_2,y_{t_2}+\hat{\phi}_{t_2})|$$

$$+\int_0^{t_1}|A[T(t_1-s)-T(t_2-s)]|(c_1\|y_s+\hat{\phi}_s\|+c_2)ds$$

$$+\int_{t_1}^{t_2}|AT(t_2-s)|(c_1\|y_s+\hat{\phi}_s\|+c_2)ds|$$

$$+\int_0^{t_1}|T(t_1-s)-T(t_2-s)|\int_0^s \alpha_{k'}(\tau)d\tau\,ds$$

$$+\int_{t_1}^{t_2}|T(t_2-s)|\int_0^s \alpha_{k'}(\tau)d\tau\,ds,$$

where $k'=k+\|\hat{\phi}\|$. The right hand side is independent of $y\in B_k$ and tends to zero as $t_2-t_1\to 0$, since g is completely continuous and the compactness of $T(t)$ for $t>0$ implies continuity in the uniform operator topology. Thus F maps B_k into an equicontinuous family of functions.

Notice that we considered here only the case $0<t_1<t_2$, since the other cases $t_1<t_2<0$ or $t_1<0<t_2$ are very similar.

It is easy to see that the family FB_k is uniformly bounded. Next, we show $\overline{FB_k}$ is compact. Since we have shown FB_k is equicontinuous collection, by the Arzela–Ascoli theorem it suffices to show that F maps B_k into a precompact set in X.

Let $0<t\le b$ be fixed and let ϵ be a real number satisfying $0<\epsilon<t$. For $y\in B_k$ we define

$$(F_\epsilon y)(t)=-T(t)g(0,\phi)+g(t,y_t+\hat{\phi}_t)+\int_0^{t-\epsilon} AT(t-s)g(s,y_s+\hat{\phi}_s)ds$$

$$+\int_0^{t-\epsilon} T(t-s)f(s,y_s+\hat{\phi}_s,\int_0^s h(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau)ds$$

$$=-T(t)g(0,\phi)+g(t,y_t+\hat{\phi}_t)$$

$$+T(\epsilon)\int_0^{t-\epsilon} AT(t-s-\epsilon)g(s,y_s+\hat{\phi}_s)ds$$

$$+T(\epsilon)\int_0^{t-\epsilon} T(t-s-\epsilon)f\left(s,y_s+\hat{\phi}_s,\int_0^s h(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau\right)ds.$$

Since $T(t)$ is a compact operator, the set $Y_\epsilon(t)=\{(F_\epsilon y)(t): y\in B_k\}$ is precompact in X for every ϵ, $0<\epsilon<t$. Moreover, for every $y\in B_k$ we

have

$$
\begin{aligned}
&|(Fy)(t) - (F_\epsilon y)(t)| \\
&\quad \le \int_{t-\epsilon}^{t} |AT(t-s)g(s, y_s + \hat{\phi}_s)|ds \\
&\qquad + \int_{t-\epsilon}^{t} |T(t-s)f\left(s, y_s + \hat{\phi}_s, \int_0^s h(s, \tau, y_\tau + \hat{\phi}_\tau)d\tau\right)|ds \\
&\quad \le \int_{t-\epsilon}^{t} |AT(t-s)g(s, y_s + \hat{\phi}_s)|ds \\
&\qquad + \int_{t-\epsilon}^{t} |T(t-s)| \int_0^s \alpha_{k'}(\tau)d\tau\, ds.
\end{aligned}
$$

Therefore, there are precompact sets arbitrarily close to the set $\{(Fy)(t) : x \in B_k\}$. Hence the set $\{(Fy)(t) : y \in B_k\}$ is precompact in X.

It remains to show that $F: C_b^0 \to C_b^0$ is continuous. Let $\{y_n\}_0^\infty \subseteq C_b^0$ with $y_n \to y$ in C_b^0. Then there is an integer r such that $\|y_n(t)\| \le r$ for all n and $t \in J$, so $y_n \in B_r$ and $y \in B_r$. By (ii) and (iii)

$$
\begin{aligned}
&f\left(t, y_{n_t} + \hat{\phi}_t, \int_0^t h(t, s, y_{ns} + \hat{\phi}_s)ds\right) \\
&\qquad \to f\left(t, y_t + \hat{\phi}_t, \int_0^t h(t, s, y_s + \hat{\phi}_s)ds\right)
\end{aligned}
$$

for each $t \in J$ and since

$$
\begin{aligned}
&\Bigg| f\left(t, y_{n_t} + \hat{\phi}_t, \int_0^t h(t, s, y_{n_s} + \hat{\phi}_s)ds\right) \\
&\qquad - f\left(t, y_t + \hat{\phi}_t, \int_0^t h(t, s, y_s + \hat{\phi}_s)ds\right)\Bigg| \\
&\quad \le 2\alpha_{r'}(t),
\end{aligned}
$$

$r' = r + \|\hat{\phi}\|$, and also g is completely continuous, by the dominated convergence theorem we have

$$
\begin{aligned}
\|Fy_n - Fy\| = \sup_{t\in J} &|[g(t, y_{n_t} + \hat{\phi}_t) - g(t, y_t + \hat{\phi}_t)] \\
&+ \int_0^t AT(t-s)[g(s, y_{n_s} + \hat{\phi}_s) - g(s, y_s + \hat{\phi}_s)]ds \\
&+ \int_0^t T(t-s)[f\left(s, y_{ns} + \hat{\phi}_s, \int_0^s h(s, \tau, y_{n\tau} + \hat{\phi}_\tau)d\tau\right) \\
&- f\left(s, y_s + \hat{\phi}_s, \int_0^s h(s, \tau, y_\tau + \hat{\phi}_\tau)d\tau\right)]ds|
\end{aligned}
$$

$$
\begin{aligned}
&\leq |g(t, y_{nt} + \hat{\phi}_t) - g(t, y_t + \hat{\phi}_t)| \\
&\quad + \int_0^b |AT(t-s)||g(s, y_{ns} + \hat{\phi}_s) - g(s, y_s + \hat{\phi}_s)|ds \\
&\quad + \int_0^b |T(t-s)||f\left(s, y_{ns} + \hat{\phi}_s, \int_0^s h(s, \tau, y_{n\tau} + \hat{\phi}_\tau)d\tau\right) \\
&\quad - f\left(s, y_s + \hat{\phi}_s, \int_0^s h(s, \tau, y_\tau + \hat{\phi}_\tau)d\tau\right)|ds \to 0, \quad \text{as } n \to \infty.
\end{aligned}
$$

Thus F is continuous. This completes the proof that F is completely continuous.

Finally the set $\zeta(F) = \{y \in C_b^0 : y = \lambda F y, \lambda \in (0,1)\}$ is bounded, since for every solution y in $\zeta(F)$ the function $x = y + \hat{\phi}$ is a mild solution of (3), for which we have proved that $\|x\|_1 \leq K$ and hence

$$
\|y\|_1 \leq K + \|\hat{\phi}\|.
$$

Consequently, by Schaefer's theorem the operator F has a fixed point in C_b^0. This means that the problem (1) has a mild solution.

Remark. If g is uniformly bounded, then $|g(t, \phi)| \leq c_2$ for $t \in J, \phi \in C$. In that case $c = M_1(\|\phi\| + c_2) + c_2 + M_2 c_2 b$ and $\hat{m}(t) = \max\{M_1 p(t), \alpha m(t)\}$. So, the mild solution exists on $[-r, b]$.

4. EXAMPLE

Consider the partial integrodifferential equation

$$
\begin{aligned}
&\frac{\partial}{\partial t}[z(y,t) - p(t, z(y, t-r))] \\
&\qquad = \frac{\partial^2}{\partial y^2} z(y,t) + q(t, z(y, t-r), \\
&\qquad \int_0^t k(t, s, z(y, s-r))ds), \quad 0 \leq y \leq \pi, \qquad t \in J, \\
&z(0,t) = z(\pi, t) = 0, \qquad t \geq 0, \\
&z(t,y) = \phi(y,t), \qquad -r \leq t \leq 0, \qquad (7)
\end{aligned}
$$

where ϕ is continuous and p, q, and k are continuous and satisfy certain smoothness conditions. Let $g(t, w_t)(y) = p(t, w(t-y))$, $h(t, s, w_s)(y) = k(t, s, w(s-y))$, and $f(t, w_t, v)(y) = q(t, w(t-y), v(y))$, $0 \leq y \leq \pi$.

Take $X = L^2[0, \pi]$ and define $A: X \to X$ by $Aw = w''$ with domain D(A)= $\{w \in X : w, w'$ are absolutely continuous, $w'' \in X$, $w(0) = w(\pi) = 0\}$. Then

$$Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \qquad w \in D(A),$$

where $w_n(y) = \sqrt{2/\pi} \sin ny$, $n = 1, 2, 3, \dots,$ is the orthogonal set of eigenvectors of A.

It is well known that A is the infinitesimal generator of an analytic semigroup $T(t)$, $t \geq 0$ in X, and is given by [12]

$$T(t)w = \sum_{n=1}^{\infty} \exp(-n^2 t)(w, w_n) w_n, \qquad w \in X.$$

Since the analytic semigroup $T(t)$ is compact, there exist constants $N \geq 1$ and $N_1 > 0$ such that $|T(t)| \leq N$ and $|AT(t)| \leq N_1$ for each $t > 0$.

Further, the function $p: J \times [0, \pi] \to [0, \pi]$ is completely continuous and uniformly bounded; that is, there exists a constant $n_1 > 0$ such that

$$\| p(t, w(t - y)) \| \leq n_1.$$

Also, the functions $k: J \times J \times [0, \pi]$ and $q: J \times [0, \pi] \times [0, \pi] \to [0, \pi]$ are measurable and there exist integrable functions $l_1, l_2: J \to [0, \infty)$ and a constant $n_2 > 0$ such that

$$\| k(t, s, w) \| \leq \alpha l_1(s) \Omega_0(\| w \|)$$

and

$$\| q(t, v, w) \| \leq l_2(t) \Omega(\| v \| + \| w \|),$$

where $\Omega_0, \Omega: [0, \infty) \to (0, \infty)$ are continuous, nondecreasing, and

$$\int_0^b \hat{n}(s) ds < \int_c^{\infty} \frac{ds}{s + \Omega_0(s) + \Omega(s)},$$

where $c = N(\| \phi \| + n_1) + n_1 + N_1 n_1 b$ and $\hat{n}(t) = \max\{N l_2(t), n_2 l_1(t)\}$.

Since all the conditions of the Theorem 3.1 are satisfied, Eq. (7) has a mild solution on $[-r, b]$.

5. APPLICATION

As an application of Theorem 3.1, we shall consider the system (1) with a control parameter such as

$$\begin{aligned}\frac{d}{dt}[x(t)-g(t,x_t)] &= Ax(t)+Bu(t)\\ &\quad + f\left(t,x_t,\int_0^t h(t,s,x_s)ds\right), \qquad t\in J,\\ x_0 &= \phi, \qquad \text{on } [-r,0],\end{aligned} \tag{8}$$

where B is a bounded linear operator from U, a Banach space, to X and $u \in L^2(J,U)$.

The mild solution is given by

$$\begin{aligned}x(t) &= T(t)[\phi(0)-g(0,\phi)]+g(t,x_t)+\int_0^t AT(t-s)g(s,x_s)ds\\ &\quad +\int_0^t T(t-s)\left[Bu(s)+f\left(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\right)\right]ds, \qquad t\in J,\\ x_0 &= \phi, \qquad \text{on } [-r,0].\end{aligned}$$

DEFINITION 5.1. System (8) is controllable to the origin on the interval J if for every continuous initial function $\phi \in C$, there exists a control $u \in L^2(J,U)$ such that the mild solution $x(t)$ of (8) satisfies $x(b)=0$.

For the controllability of neutral systems one can refer to the paper [4] and the references cited therein. To establish the controllability result we need the following additional hypotheses:

(x) The linear operator $W\colon L^2(J,U)\to X$, defined by

$$Wu = \int_0^b T(b-s)Bu(s)\,ds,$$

has an induced inverse operator $\widetilde{W}^{-1}$ which takes values in $L^2(J,U)/\ker W$ and there exists a positive constant M_3 such that $|B\widetilde{W}^{-1}| \le M_3$.

(xi)

$$\int_0^b \hat{m}(s)ds < \int_c^\infty \frac{ds}{s+\Omega_0(s)+\Omega(s)},$$

where

$$c = \frac{1}{1-c_1}[M_1(\|\phi\|+c_1\|\phi\|+c_2)+c_2+M_2c_2b+M_1Nb],$$

$$\hat{m}(t) = \max\left\{\frac{M_2 c_1}{1-c_1}, \frac{M_1}{1-c_1} p(t), \quad \alpha m(t)\right\}, \quad \text{and}$$

$$N = M_3\Big[M_1(\|\phi\| + c_1\|\phi\| + c_2) + c_1\|x_b\| + c_2 + M_2\int_0^b (c_1\|x_s\| + c_2)ds$$
$$+ M_1\int_0^b p(s)\Omega\Big(\|x_s\| + \int_0^s \alpha m(\tau)\Omega_0(\|x_\tau\|)d\tau\Big)ds\Big].$$

THEOREM 5.1. *If the hypotheses* (i)–(viii) *and* (x)–(xi) *are satisfied, then the system* (8) *is controllable.*

Proof. Using the hypothesis (x), for an arbitrary function $x(\cdot)$, define the control

$$u(t) = -\widetilde{W}^{-1}\Big[T(b)[\phi(0) - g(0,\phi)] + g(b, x_b) + \int_0^b AT(b-s)g(s, x_s)ds$$
$$+ \int_0^b T(b-s)f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)ds\Big](t).$$

We shall show that when using this control the operator $\Phi\colon C_b^0 \to C_b^0$ defined by

$$(\Phi x)(t) = T(t)[\phi(0) - g(0,\phi)] + g(t, x_t) + \int_0^t AT(t-s)g(s, x_s)ds$$
$$+ \int_0^t T(t-s)\Big[Bu(s) + f(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau)\Big]ds, \qquad t \in J,$$
$$= \phi(t), \qquad t \in [-r, 0]$$

has a fixed point. This fixed point is then a solution of Eq. (8). Substituting $u(t)$ in the above equation we get $(\Phi x)(b) = 0$, which means that the control u steers system (8) from the given initial function ϕ to the origin in time b, provided we can obtain a fixed point of the nonlinear operator Φ. The remaining part of the proof is similar to Theorem 3.1 and hence it is omitted.

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